UNIT - III.

# Central Difference Interpolation Formula (For equal Intervals).

We knows that, central difference operator,

$$\delta = E^{1/2} - E^{-1/2}, \quad \text{Also} \quad \delta = \Delta E^{-1/2}.$$

The Newton's interpolation formula which are used only for interpolation near the beginning and the end of the table. They are not applicable to interpolate near the central value. To get more accurate results near the middle value of the table, we will obtain a more suitable formula which utilises difference close to the middle value of the table. Such formula are named as central difference interpolation formula.

## Gauss's Forward Interpolation Formula.

By the Newton's Gregory forward interpolation formula,

We have

$$y(x) = y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots \rightarrow ①$$

where

$$u = \frac{x - x_0}{h}.$$

Now,

$$\Delta^2 y_0 = \Delta^2 E y_{-1} = \Delta^2(1+\Delta) y_{-1} = \Delta^2 y_{-1} + \Delta^3 y_{-1} \rightarrow (i)$$

$$\Delta^3 y_0 = \Delta^3 E y_{-1} = \Delta^3(1+\Delta) y_{-1} = \Delta^3 y_{-1} + \Delta^4 y_{-1} \rightarrow (ii)$$

$$\Delta^4 y_0 = \Delta^4 E y_{-1} + \Delta^5 y_{-1}. \rightarrow (iii)$$

$\text{III}^{\text{ly}}$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2} \quad \text{and so on.}$$

Substituting the values of $\Delta^2 y_0$, $\Delta^3 y_0$, ... in ①, We have

$$y(x) = y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \frac{u(u-1)(u-2)}{3!}$$

$$(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{u(u-1)(u-2)(u-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

$$= y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \left[\binom{u}{2} + \binom{u}{3}\right]\Delta^3 y_{-1} + \left[\binom{u}{3} + \binom{u}{4}\right]\Delta^4 y_{-1} + \cdots$$

$$= y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-1} + \binom{u+1}{5}\Delta^5 y_{-1} + \cdots$$

$$= y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\left[\Delta^4 y_{-2} + \Delta^5 y_{-2}\right]$$
$$- \binom{u+1}{5}\left[\Delta^5 y_{-2} + \Delta^6 y_{-2}\right] + \cdots$$

$$= y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} + \left[\binom{u+1}{4} + \binom{u+1}{5}\right]$$
$$\Delta^5 y_{-2} + \cdots$$

$$y(x_0 + uh) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-2} +$$
$$\binom{u+2}{5}\Delta^5 y_{-2} + \cdots \quad \to (2)$$

Where
$$\binom{u}{r} = \frac{u(u-1)(u-2)(u-3)\cdots(u-r+1)}{r!}$$

This formula is known as Gauss's forward interpolation formula.

Note:
1) This formula involves odd differences below the central line $(x=a)$ and even differences on the line.
2) Taking the central line and the next line, ~~from~~ ~~the table~~ I, we have the differences occuring in the formula.

- Central line $\quad y_0 \quad \Delta^2 y_{-1} \quad \Delta^4 y_{-2} \quad \Delta^6 y_{-3}$

  next line $\quad \Delta y_0 \quad \Delta^3 y_{-1} \quad \Delta^5 y_{-2}$

3. The formula can be written easily with the help of the following table.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Coefficients | $1$ | $\binom{u}{1}$ | $\binom{u}{2}$ | $\binom{u+1}{3}$ | $\binom{u+1}{4}$ | $\binom{u+2}{5}$ | ... |
| Differences | $y_0$ | $\Delta y_0$ | $\Delta^2 y_{-1}$ | $\Delta^3 y_{-1}$ | $\Delta^4 y_{-2}$ | $\Delta^5 y_{-2}$ | ... |

4. This formula is useful when $u$ lies between 0 & 1.

Pb. Apply Gauss forward formula to obtain $f(x)$ at $x = 3.5$ from the table.

| | | | | |
|---|---|---|---|---|
| $x$ : | 2 | 3 | 4 | 5 |
| $f(x)$ : | 2.626 | 3.454 | 4.784 | 6.986 |

Sol. Take $x = 3$ as the origin; here $h = 1$.

$$u = \frac{x - x_0}{h} = \frac{3.5 - 3}{1} = 0.5$$

3.5 -

Let us form the central diff. table.

| $x$ | $u$ | $y = f(x)$ | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ |
|---|---|---|---|---|---|
| 2 | -1 | 2.626 | | | |
| | | | 0.828 | | |
| | | | | $\boxed{0.502}$ | |
| 3 | 0 | $\boxed{3.454}$ | | | $\boxed{0.37}$ |
| | | | $\boxed{1.330}$ | | |
| 4 | 1 | 4.784 | | 0.872 | |
| | | | 2.202 | | |
| 5 | 2 | 6.986 | | | |

The values required are enclosed by rectangles.

$$y(u = 0.5) = y_0 + \binom{u}{1}\Delta y_0 + \binom{u}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \dots$$

$$= 3.454 + 0.5(1.330) + \frac{0.5(-0.5)}{2}(0.502) + \frac{1.5(0.5)(-0.5)}{6}(0.37)$$

$$= 3.456 + 0.665 - 0.06275 - 0.023125$$

$$= 4.033125$$

## II Gauss's Backward Interpolation Formula.

Starting with Newton-Gregory forward interpolation formula at $x = x_0$, we have

$$y(x) = y(x_0 + uh) = y_0 + \Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots \rightarrow (**)$$

where $u = \frac{x - x_0}{h}$.

Since $\Delta y_0 = \Delta E y_{-1} = \Delta(1+\Delta) y_{-1} = \Delta y_{-1} + \Delta^2 y_{-1}$.

$|||^{ly}$ $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$.

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2} \text{ etc.}$$

Substituting the values of $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ in $(**)$ we get

$$y(x) = y(x_0 + uh) = y_0 + u(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{u(u-1)}{2!}\left[\Delta^2 y_{-1} + \Delta^3 y_{-1}\right]$$
$$+ \frac{u(u-1)(u-2)}{3!}\left[\Delta^3 y_{-1} + \Delta^4 y_{-1}\right]$$
$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\left[\Delta^4 y_{-1} + \Delta^5 y_{-1}\right] + \cdots$$

$$= y_0 + u\Delta y_{-1} + \left[\binom{u}{1}+\binom{u}{2}\right]\Delta^2 y_{-1} + \left[\binom{u}{2}+\binom{u}{3}\right]\Delta^3 y_{-1} + \left[\binom{u}{3}+\binom{u}{4}\right]\Delta^4 y_{-1} + \cdots$$

$$= y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-1} + \binom{u+1}{4}\Delta^4 y_{-1} + \cdots$$

$$= y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\left[\Delta^3 y_{-2} + \Delta^4 y_{-2}\right] + \binom{u+1}{4}\left[\Delta^4 y_{-2} + \Delta^5 y_{-2}\right] + \cdots$$

$$= y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2} + \binom{u+2}{4}\Delta^4 y_{-2} + \cdots$$

$$y(x) = y(x_0 + uh) = y_0 + \binom{u}{1}\Delta y_{-1} + \binom{u+1}{2}\Delta^2 y_{-1} + \binom{u+1}{3}\Delta^3 y_{-2} + \binom{u+2}{4}\Delta^4 y_{-2} + \cdots$$

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This is called Gauss's backward ~~forward~~ formula of interpolation.

Note:

1. This formula involves odd differences above the central line and even differences on the central line.
2. Taking the central line and the previous line of the table 1, we have the differences occurring in the formula.

Previous line : $\Delta y_{-1}$    $\Delta^3 y_{-2}$    $\Delta^5 y_{-3}$

Central line : $y_0$    $\Delta^2 y_{-1}$    $\Delta^4 y_{-2}$    $\Delta^6 y_{-3}$

3. The formula can be easily written with the help of the following table.

| | | | | | | |
|---|---|---|---|---|---|---|
| Coefficient : | 1 | $\binom{u}{1}$ | $\binom{u+1}{2}$ | $\binom{u+1}{3}$ | $\binom{u+2}{4}$ | $\cdots$ |
| Differences : | $y_0$ | $\Delta y_{-1}$ | $\Delta^2 y_{-1}$ | $\Delta^3 y_{-2}$ | $\Delta^4 y_{-2}$ | |

4. This backward formula is useful when $u$ lies between $-1$ and $0$.